



Attribution of Climate Change to the extreme snow falls of mazandaran province (stochastic climate modeling using lorenz-63)

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Article Info	ABSTRACT
Article type: Research Article Article history: Received 2022/06/26 Received in revised 2022/11/20 Accepted 2022/12/15 Published 2022/12/16 Published online 2025/12/22 Keywords: Attribution, stochastic process, Lorenz-63, Ensemble Kalman Filter (EnKF), Causal (counterfactual) theory.	The aim of this study was to attribute climate change to the occurrence of extreme snowfall in Mazandaran Province, located in northern Iran along the southern shores of the Caspian Sea and in the vicinity of the Alborz Mountains. This region is not typically prone to extreme snowfall; therefore, such events can cause significant damage to the region's infrastructure. The study focused on the winter season (DJF) during the period from 1987 to 2017. The corresponding calculations were performed for two parallel counterfactual worlds: one without external forcing and one factual world with external forcing, within the framework of stochastic climate modeling. This was achieved using the chaotic dynamical three-dimensional Lorenz-63 (Lorenz-63) model. The two study worlds were defined on the basis of the deterministic LM-63 and stochastic SLM-63 climate models, respectively. The Fokker–Planck equation was used to implement the temporal evolution of the probability density function (PDF) within the model. Conditional probability and the Bayesian framework constitute the methodological foundations of this study. The model belongs to the class of state-space models and employs Bayesian recursive estimation. Together, these elements form the basis of the Ensemble Kalman Filter (EnKF), a nonlinear filtering approach applied to the nonlinear dynamical model used in this study. An effort was made to address all relevant aspects of the problem on the basis of sound mathematical, epistemological, and physical foundations. All computations were performed using GIS, MATLAB, Mathematica, and Maple. Pearl's (2000) causal theory was subsequently employed as evidence for verifying the overall process. The final results showed that the occurrence of extreme snowfall in Mazandaran Province is attributable to the climate forcing defined in this study, with a probability of necessity (PN) of 0.8978, a probability of sufficiency (PS) of 0.1942, and a probability of necessity and sufficiency (PNS) of 0.4519.

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Introduction

One of the main reasons for studying rare events is that they are often associated with undesirable incidents with potentially severe environmental, social, financial, infrastructural, or technological impacts (Hannart et al., 2016). Regular weather patterns can become irregular and more intense, followed by potentially devastating consequences. The fundamental questions are therefore: Is such an event a natural component of the climate system, or is it influenced by human activities?

Extreme snowfall in a region such as Mazandaran Province, Iran, is not a regular wintertime event. Although snowfall can occur in the region, it is relatively rare in many parts of the province. In the mountainous areas in the south of the province, however, snowfall is a common form of winter precipitation. There is no universally accepted definition of extreme snowfall; however, the American Meteorological Society (AMS) provides definitions based on snowfall amounts, with thresholds depending on the particular context (Mach et al., 2014). Such thresholds are also strongly influenced by the geographical and geomorphological characteristics of the region under consideration.

The study of extreme events is an active area of research involving several different approaches, particularly in the context of detection and attribution. Like other scientific fields, extreme-event research relies on fundamental definitions, propositions, theorems, and mathematical concepts. Mathematics provides a formal language for describing physical systems; consequently, the detection and attribution of climate-change-related extreme events also require a mathematically consistent framework. A physically meaningful problem should first be formulated correctly in physical terms, followed by a mathematically consistent formulation and an appropriate solution method. This general approach is fundamental to the analysis of physical systems, including the climate system. In atmospheric sciences, direct laboratory experimentation on the climate system is not possible at the planetary scale; consequently, numerical models serve as essential experimental and investigative tools (Andrews, 2010). Establishing an appropriate climate and weather model is therefore of primary importance in the study of extreme-event attribution.

The climate system is a highly complex physical system (Mach et al., 2014). It is also an open, extended system rather than a closed system (Kardar, 2007), and therefore continuously interacts with its subsystems and with its planetary environment. Consequently, a simple closed Hamiltonian dynamical-system formulation is not sufficient to describe the complete climate system (Kardar, 2007). Current developments in climate modeling increasingly incorporate physical and mathematical foundations beyond the traditional classical formulations, including approaches based on nonlinear dynamics, stochastic processes, nonequilibrium systems, and other modern physical and mathematical frameworks. The fundamental discussions presented in this study are primarily motivated by the problem of climate-system modeling, although the underlying framework has a broader applicability. This perspective is maintained throughout the study because the proposed approach to extreme-event attribution requires a consistent physical and mathematical foundation. The climate system exhibits nonlinear, chaotic, dissipative, nonequilibrium, and probabilistic behavior. Moreover, uncertainties arise from unresolved processes, imperfect observations, parameterizations, and external variability. These characteristics motivate the use of stochastic dynamical descriptions in addition to deterministic formulations. The specific event considered in this study—the extreme snowfall event in Mazandaran Province—is itself a rare event. Consequently, the attribution problem is formulated within a framework that explicitly represents uncertainty and stochasticity. The theoretical foundation is therefore developed first, and this foundation is subsequently used to establish

the methodology presented in Section 3. From a methodological perspective, the proposed framework can be classified within the broader family of data-assimilation approaches.

Section 3 develops the methodology on the basis of the mathematical and physical foundations established earlier. It introduces a low-dimensional three-dimensional deterministic model used to construct the counterfactual and factual worlds and then incorporates additive white noise to formulate their stochastic counterparts. These two model configurations provide the dynamical framework within which the attribution problem is investigated.

The concepts of causality discussed by Nietzsche (1901/1968) and Pearl (2000) are brought together in Sections 2 and 3 to develop the concept of causal attribution of extreme events. The objective is to establish a coherent connection between the theoretical foundations, the mathematical formulation, the dynamical modeling, the data-assimilation procedure, and the causal-attribution framework, and to apply this integrated approach to the extreme snowfall event in Mazandaran Province.

2.Theoretical background and literature

2.1. Basic properties of climate system as physical system

According to the IPCC (2014), the climate system is a highly complex system comprising five principal components: the atmosphere, the hydrosphere, the cryosphere, the lithosphere, and the biosphere. Interactions among these components govern the functioning and evolution of the climate system (Mach et al., 2014). The climate system is a dynamical system because it evolves over time. This evolution is influenced by its internal dynamics (Palmer, 1999; Werndl, 2016) as well as by external forcings. These forcings include volcanic eruptions, variations in solar activity, and anthropogenic influences. Among the most prominent human-induced changes are perturbations in atmospheric composition and alterations in land use (Mach et al., 2014; Palmer, 1999; Wallace & Hobbs, 2006).

The study of such a chaotic and complex system requires an explicit characterization of uncertainties, which "arise not only from initial conditions and forcing scenarios, but also from model formulation" (Knutti & Rugenstein, 2015; Williams et al., 2013). In this context, climate scientists themselves recognize the need for further work on the theoretical foundations of climate science (Williams et al., 2013). According to Hegerl et al. (2010), the predictive capability of the climate system as a whole is constrained, among other factors, by limitations in the current formulation and understanding of its governing physical principles. Based on Hegerl et al. (2010) and Williams et al. (2013), one potentially productive way of addressing this problem is to define appropriate mathematical and physical concepts for the study of different branches of climate science.

From the perspective of dynamical systems, Palmer (1999) described climate change in terms of changes in the climate-system attractor resulting from external forcing. Although many prominent definitions of climate change are formulated in statistical or mathematical terms, Hegerl et al. (2010), from a broader perspective, described climate change as "any multi-decadal or longer alteration in one or more physical, chemical and/or biological components of the climate system". Thus, the concept of climate change is closely related to the concept of a climate regime (Williams et al., 2013). Climate feedback is an internal process within the climate system that can amplify or diminish the initial effects of external forcing or internal variability (Wallace & Hobbs, 2006). In this context, climate sensitivity can be expressed in terms of feedback factors, as represented by the following relationship:

$$s = \frac{dT}{dF} = \frac{\partial T}{\partial F} + \sum_i \frac{\partial T}{\partial Y_i} \frac{\partial Y_i}{\partial F} \quad (2.1)$$

These sorts of definitions are able to offer an appropriate general mathematical characterization of climate sensitivity. As noted above, they also describe the response of the climate system to additional radiative forcing. However, climate sensitivity may depend on the state and conditions of the climate system (Bindoff et al., 2013; Katzav & Parker, 2015). Consequently, the characterization of a particular climate state remains subject to uncertainty and scientific debate. More generally, it appears that different forms of dependence can be identified across the various concepts used to describe the climate system and its behavior (Knutti & Rugenstein, 2015).

According to the IPCC (2014), much of our knowledge about climate variability is derived from experiments using numerical models (Mach et al., 2014). The basis of the inference in this study is the consistency of the boundary conditions between the historical observations and the model simulations, together with the consideration of variations relative to seasonal variability and the climatological mean on a year-by-year basis.

Climate variability can broadly be separated into two components: internal variability and externally forced variability (Katzav & Parker, 2015; Knutti & Rugenstein, 2015). Internal variability refers to naturally occurring variations within the climate system, whereas externally forced variability refers to changes associated with external forcings, including anthropogenic forcing. However, in approaches such as the fingerprint method—which is a key approach in climate-change attribution—these components are generally represented as contributions to the total observed climate variability (Hayhoe et al., 2017; Katzav & Parker, 2015; Miller et al., 1999). In this framework, the observed variability is interpreted in terms of the combined effects of internal variability and external forcing.

Within such attribution frameworks, internal variability is commonly characterized using simulations in which the relevant external forcing is held constant or absent, thereby providing a representation of the climate variability that can arise from the internal dynamics of the climate system (Jézéquel et al., 2018; Trenberth et al., 2015).

Detection is defined as "the process of demonstrating that climate or a system affected by climate has changed in some defined statistical sense without providing a reason for that change" (Stone & Allen, 2005). A change is considered to be detected in observations when the likelihood that the observed change could have occurred by chance as a result of internal variability alone is sufficiently small (IPCC, 2014; Knutti & Rugenstein, 2015; Shepherd, 2016).

Attribution, in contrast, is defined as "the process of evaluating the relative contributions of multiple causal factors to a change or event with an assignment of statistical confidence" (Ornes, 2018). Thus, while detection addresses whether a statistically significant change has occurred, attribution addresses the extent to which specific causal factors contributed to that observed change.

2.2 complexity of the climate system

However, in the aforementioned concepts and propositions, which are largely based on a deterministic framework for climate studies, modeling, detection, and attribution, the role of complexity in the climate system is not sufficiently represented (Allawala & Marston, 2016; Miller et al., 1999; Palmer, 1999). A mathematical description and analysis of the climate system requires a sound physical foundation, since the physical processes within the system initiate and govern the phenomena and observations under investigation.

The climate system is undoubtedly highly complex; however, establishing more physically consistent definitions can provide an alternative perspective to the traditional viewpoint, which has been strongly influenced by classical dynamical-system formulations based on Hamiltonian concepts (Lucarini et al., 2014). Such classical frameworks are commonly associated with conservation laws, equilibrium states, and deterministic evolution. In contrast, modern approaches emphasize nonequilibrium dynamics, dissipative processes, stochasticity, and intrinsic randomness.

The climate system is also characterized by both chaotic and complex behavior. In this context, chaos refers to the emergence of highly sensitive and sometimes abrupt changes resulting from nonlinear dynamical interactions, whereas complexity refers to the presence of multiple interacting processes and structures whose complete description may be practically inaccessible.

From a mathematical perspective, the fundamental concept of chaos is commonly defined through sensitivity to initial conditions in systems of differential equations. Small differences in initial states can lead to exponentially diverging trajectories over time, resulting in limited long-term predictability. This behavior is quantified through Lyapunov exponents, which measure the rate of separation of initially nearby trajectories during the temporal evolution of a dynamical system (Franzke et al., 2015; Hannart et al., 2016).

2.3 Epistemological Roots

Epistemologically, this study is based on causal counterfactual theory for the attribution of weather and climate events. Although this approach has its roots in traditional theories of cause and effect (Hannart et al., 2016; Mazandaran: Geography and History, 2008; Nietzsche, 1901/1968), the present study addresses the problem of clearly defining and interpreting causal factors by incorporating ideas associated with Nietzsche into its epistemological and scientific evaluation framework (Nietzsche, 1901/1968).

2.4 the region

Mazandaran Province is located along the southern coast of the Caspian Sea and adjacent to the central Alborz Mountains. It is one of the northern provinces of Iran and is situated in the central-northern part of the country. The province covers an area of approximately 24,000 km², or more precisely 23,833 km² (9,202 mi²), and extends approximately between 35°47' and 36°35' N latitude and 50°34' and 54°10' E longitude. To the north, it borders the Caspian Sea, while to the south it extends toward the Alborz Mountains. The province has a population of approximately 3,073,943 people (Encyclopaedia Britannica, 2007; Iran Daily, 2007; Nicholls & Toumi, 2014; Scheffers & Kelletat, 2016).

The presence of the Caspian Sea has a substantial influence on surface air temperatures in the study region. Because of the sea's thermal inertia, the Caspian Sea tends to moderate seasonal temperature variations, contributing to lower temperatures during summer and relatively higher temperatures during winter. These temperature differences influence surface cyclogenesis through their associated pressure signals (Mackwell et al., 2013; National Research Council, 2005).

The region is characterized by diverse natural landscapes, including coastal plains, forests, and the snow-capped Alborz Mountains. The Alborz range forms a large natural barrier enclosing much of the coastal strip and adjacent plains. The dominant wind directions are generally from the north and east (Encyclopaedia Britannica, 2007; Nicholls & Toumi, 2014).

Snowfall occurs frequently in the mountainous areas, where snowmelt and precipitation contribute to numerous rivers and streams that flow toward the Caspian Sea. In contrast to

many other parts of Iran, the region has an extensive network of perennial and seasonal waterways. These rivers and streams divide parts of the landscape into numerous valleys. The regional climate is generally characterized as moderate and subtropical, with an average temperature of approximately 25 °C during summer and about 8 °C during winter. Although heavy snowfall can occur in the mountainous areas during winter, snowfall is relatively rare at sea level and across the lower coastal plains. Figure 1 presents the relief map of the study region (Encyclopaedia Britannica, 2007; Scheffers & Kelletat, 2016).

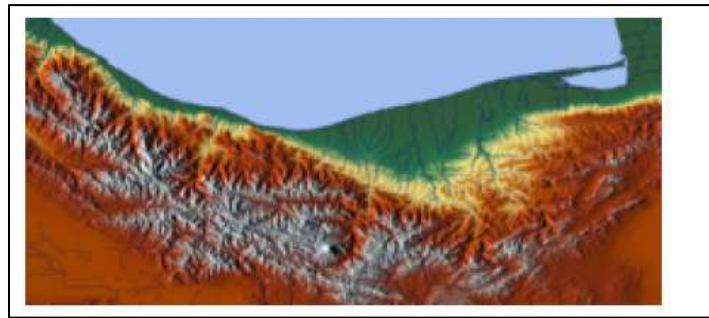


Fig 1. The relief map of Mazandaran province

Therefore, the regional climate is influenced by latitude, elevation above sea level, the distance between the Alborz Mountains and the Caspian Sea, the dry and cold barren regions of Turkmenistan to the southeast, local atmospheric circulations such as sea breezes and prevailing winds, and the region's diverse vegetation cover (Mackwell et al., 2013; National Research Council, 2005).

3. Methods and Materials

3.1 Data

Since this study takes a general view of the considered problem, general types of parameters have been selected. These data include mean temperature, net radiation flux, and snowfall data as snow equivalent water from the NCEP-NCAR reanalysis (2) data series. The study period, i.e., the time interval, has been selected as 30 years of DJF from 1987 to 2017. The net radiation flux is calculated as radiation at the surface. Then, for simplicity and to save time, the ground level was selected. The ground level was chosen because the parameter of interest, temperature (air), is sensed up to 2 meters above the ground, while the net radiation flux is calculated from four series of data using the equation for net radiation flux (R_n) (Katzav & Parker, 2015; Landau & Lifshitz, 1970): $R_n = SW\downarrow - SW\uparrow + LW\downarrow - LW\uparrow$.

This selection was made because the net radiation at the surface is the result of a "combination of different radiative processes" and directly affects the temperature (air), which is also one of the main drivers of the climate system and atmosphere (Landau & Lifshitz, 1970; Majda et al., 2001). Another reason is that these are general parameters and are suitable for the top-down view of the study. The region had to be on a larger scale since the resolution of the raster data is 1.75×1.75 meters. Therefore, the region was selected to be within N 35–45 and E 25–65, as shown in Figures 2 below. This was done for three corresponding parameters. However, having a global scale is more efficient than having a box, but this was done for simplicity and to have lower computation cost.

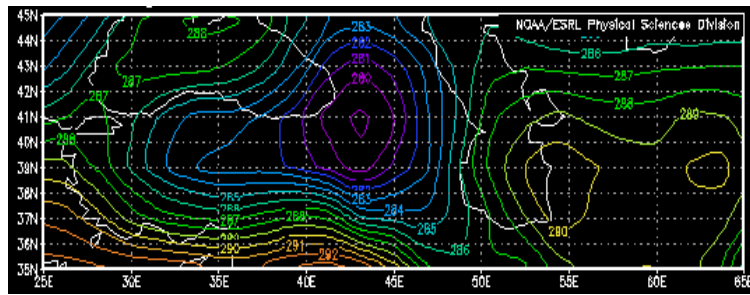


Fig. 2. Daily mean temperature (air) (K) for coordinates 25°E–65°E and 35°N–45°N at the 2.00 level. Time range: January 1, 1987, to January 1, 2017.

Fig. 2. Max = 295.977 K and min = 278.727 K. (NCEP/DOE AMIP-II Reanalysis). This map is presented here as evidence of the study region and also to show the total distribution of the selected parameter (air) over the entire period. Mazandaran Province, as indicated by the title of the study, is located in the southern part of the Caspian Sea.

Because of technical problems with the Mazandaran weather stations, especially in providing comprehensive temporal and spatial coverage of the whole region, the study was forced to use satellite data for this case study. Moreover, because of the simplicity of access and the availability of a longer time interval, the parameters were selected from NCEP, most of them from NOAA satellites.

3.2 Method

3.2.1 The Basis of the Method

According to Landau and Lifshitz (1970), in their book *Statistical Physics*, relaxation time is "the time within which the system reaches its statistical equilibrium". Since the climate system is an open system, its relaxation time cannot be defined easily. This is one of the reasons for selecting two parallel worlds instead of a fairly long time, such as 100 years, and then dividing it into two periods. This study is conducted in two parallel worlds: factual and counterfactual. They are considered and compared instead of using a long period of unseen data, which are prepared more on the basis of mathematical functions and relations rather than on real observations. In the common approach, it is assumed that, according to the law of large numbers, in a sufficiently long time the system reaches its equilibrium, i.e., a uniform spread of its particles. However, this is not always the case, as Boltzmann proposed in his box model of the universe, which states that equilibrium is not the final fate of the system; rather, recurrences occur over and over again, which are also called Poincaré recurrences, over a sufficiently long time. Consequently, this is a universe with random fluctuations. So, predicting from the fluctuating random past raises the question: what is the most likely event that formed out of chaos? However, in the new theory, there is no reversal at all; things happen at once and then go on, and so forth (Hannart et al., 2016; Lorenz, 1968; Lucarini et al., 2016; Miller et al., 1999).

Extreme snow precipitation for a region with a moderate climate is a rare event. Therefore, it belongs to the realm of probability, not frequency. According to Hannart et al. (2016), the first step is to find a "mathematical framework (naturally a probabilistic framework) where the meaning of rare event becomes precise." Then a suitable notion is needed. Therefore, it is natural to set up a probability space χ fitted out with a σ -algebra, β , of events and a probability measure P designed to quantify the probability of the occurrence of any event $A \in B$. The Bernoulli distribution is a discrete probability distribution whose PMF is equal to $P(X = 0) = 1 - p$ and $P(X = 1) = p$, and it has a binomial distribution $B(1, p)$, where 0 represents a failure and 1 represents a success. Although, according to Hannart et al. (2016),

we can call it $y = 1$ and $y = 0$. Here, Y is defined as the result of an observed extreme situation in excess of a threshold (u) of a relevant climate index (Z), both to a large extent. According to Stone and Allen (2005), evaluating these extents is based on external forcing; mathematically, $f \in F$ has $p(Y)$. This is the starting point for factual (with external forcing, so natural) and counterfactual (without external forcing, so not natural) conditions in extreme event attribution studies. Therefore, if $p_1 = P(Y = 1 | F = 1)$ indicates factual conditions, then $p_0 = P(Y = 1 | F = 0)$ specifies counterfactual conditions. This by itself defines the FAR (Fraction of Attributable Risk), $FAR = 1 - p_0/p_1$ (Hannart et al., 2016; Nietzsche, 1901/1968; Trenberth et al., 2015).

The fraction specifies the amount of likelihood for an event that is attributable to the external forcing (F). Stott et al. (2004) interpreted the causal factors of European summer temperature anomalies based on FAR and its uncertainty, associated with model and sampling errors, as there exists a very high likelihood that anomalies greater than a threshold of 1.6°C are attributable to human-induced external forcing (Hannart et al., 2016; Nietzsche, 1901/1968; Williams et al., 2013). This conventional framework stems from the standard theory of causality. Although there exist several definitions of causation depending on counterfactual logic, Pearl (2000) and Hannart et al. (2016) note that it is still based on the conditional probability of the 18th century but has simply been extended within the probabilistic and graph framework of causal counterfactual theory. Actually, this paves the way for the application of counterfactual methodology to actual datasets and then reliable causal inference (Knutti & Rugenstein, 2015; Roy & Musielak, 2007). According to Nietzsche (1901/1968), there exists a conspicuous shortage in this context in the current literature. For our aim (attribution of extreme snowfalls of Mazandaran Province in a near-real-time view), we need two worlds to compare the probability of occurrence. We call them factual and counterfactual, as usual in this context (Nietzsche, 1901/1968): the former with external forcing and the latter without. This process is performed "in an ensemble of model simulations representing the observed climate condition," which simulate the probability of occurrence of the event in the factual world — the real world — with the probability of occurrence of the same event in the parallel ensemble of model simulations, which is the non-real alternative world. "Ensemble," in a word, means a bunch or a group. Here, it means a set of simulated climate conditions. Its roots go back to statistical physics, and especially to classical statistical mechanics and the theory of canonical ensembles, and also to its usage for the Gibbs ensembles (Lorenz, 1968; Roy & Musielak, 2007; Stone & Allen, 2005).

3.2.2 A Brief Description of the Model

The system of equations in (3.1) is a 3-D dynamical system. Originally, the Lorenz equations were derived by Saltzman (1962) as a truncated model of thermal convection in a box.

$$dx = \sigma(y - x)dt \quad dy = (\rho x - y - xz)dt \quad dz = (xy - \beta z)dt \quad (3.1)$$

For more information, see Gardiner (1997), Ghys (2013), Lorenz (1963), Palmer (1999), and Roy and Musielak (2007).

As we said before, we have two worlds to consider: Factual and Counterfactual. Also, we defined these two worlds by the Lorenz 3-D model of '63, and then said that these two modeled worlds must be computed numerically because of our long time of ensemble simulations. Now let us turn back to that point and continue with the more fundamental and practical view (Lorenz, 1963). The random noises we added to our three ordinary differential equations were Independent and Identically Distributed (i.i.d.) with zero mean, which is dubbed white noise. White noise is a time series with the same variance of its variables and also with zero correlation with other values in the series (Hannart et al., 2016).

This roots in the randomness of the process of the study, which is mentioned before as well. Here we say that actually this is the well-known Brownian motion, from which almost all of this sort of process and framework come, which is also known as the Wiener process: the former in the microscale of the matter and the latter in probability theory (Chekroun et al., 2011; Ghys, 2013). The Fokker–Planck equation gives the time evolution of the PDF of a system. There exists a connection between the FPE and SDE, which is now made by adding random white noises to the former ODE. This is because $f(y(t))$ has a conditional probability density, $p(y, t | y_0, t_0)$. This PDF is commonly calculated as the numerical solution of the associated FPE. In fact, it is the mean over Ω of the sample measures obtained for each realization ω of the corresponding noises of the related SDE. As the essence of the framework is on the conditional probability, the stochastic process explained by it, which satisfies the corresponding FPE, is actually identically equal to SDEs of the Itô calculus (Chekroun et al., 2011; Gardiner, 1997).

For this purpose and the aim of this study, we must consider the external forcing and its intensity (Katzav & Parker, 2015; Nietzsche, 1901/1968). Therefore, the previous chaotic dynamical systems must accept some noises. These random noises are supported by the random attractor of the stochastic Lorenz model (SLM), which can be represented as the following three SDEs:

$$\begin{aligned} dx &= \sigma(y - x)dt + \delta_1 v_x dw \\ dy &= \rho(x - y - xz)dt + \delta_2 v_y dw \\ dz &= (xy - \beta z)dt + \delta_3 v_z dw. \end{aligned} \tag{3.2}$$

Then the previous deterministic model is perturbed by linearly multiplicative noise in the Itô sense, with W_t a Wiener process, i.e., the stochastic forcing, which is a stationary white noise process as mentioned before, and $v > 0$ the noise intensity (0.5 in this study). The $(\delta_1, \delta_2, \delta_3)$ is identified by equal values of $\sqrt{(1/2)}$. The other parameter values are the same as the original LM-63 (Hannart et al., 2016; Hilborn, 2000; Kalnay, 2003; Van den Bossche, 2013). Consequently, this leads us to our factual world with its corresponding random noises. There are several analytical and numerical methods to solve stochastic differential equations. However, according to Hannart et al. (2016), Lucarini et al. (2016), and Van den Bossche (2013), the traditional and most important are the Itô and Stratonovich calculi. These are also called Itô SDEs and Stratonovich SDEs. All of these sorts of studies seek to gain a numerically stable model by adding some damping, as in this case study (Evensen, 2009; Lucarini et al., 2016). According to Hannart et al. (2016), using data assimilation methods in the evolving realm of stochastic climate models is a frontier of this sort of study, as "this increases the ensemble spread," which by itself gives better adjustment between observations and model forecasts. Although the Kalman filters are among the most used, for the aim of this study and also for gaining more exact results, the nonlinear Ensemble Kalman Filter is used here among other KFs. It has a strong background in the literature of the study, as it is based on the Bayesian framework and conditional probability and also on the hidden Markov process (Evensen, 2009; Pchelintsev, 2014). As mentioned before, KFs belong to the data assimilation techniques. Thus, it is better to give a brief overview of the issue as the kernel of the study.

3.2.3 The Ensemble Kalman Filter (EnKF)

This method was published for the first time in Evensen (1994). Its significance lies in the fact that here, the true covariance matrix in the KF is replaced by the sample covariance

matrix computed from the ensemble. This is in accordance with our dynamical model with its Hamiltonian and canonical distribution of its vectors. The EnKF tolerates an ensemble member or an empirical number of samples. As in the case of the canonical ensemble in classical statistical mechanics, here the size of an ensemble is chosen to be much smaller than the number of states needed in large-scale dynamical systems. Moreover, since the EnKF is an approximation using the Monte Carlo method, the selection of these ensembles is random and not deterministic. In fact, these ensemble members are samples from the PDF of the true state. This last property completely answers our need for a stochastic computation (Jacobson, 2005; Lucarini et al., 2014).

$$\begin{aligned} \mathbf{x}_{(k+1)} &= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{k}) + \mathbf{w}_k \\ \mathbf{y}_k &= \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{k}) + \mathbf{v}_k \end{aligned} \quad (3.3)$$

For the algorithm of the EnKF, please refer to Jacobson (2005) and Lucarini et al. (2014); it can be represented as below, which provides the final clarification.

4. Results and Discussion

4.1 Initial Calculations

In the nonlinear system of three ODEs, the Lorenz-63 model yielded the required counterfactual world for us. The reason is that the forcing we need as a driver exists explicitly in the model. In addition, the model is derived and truncated from the Navier–Stokes equations. This was accomplished with the initial condition provided by the prepared PDF of the parameters of mean temperature, net radiation flux, and snowfalls (snow equivalent water) for the DJF of 1978 to 2017. However, for an initial test, it is preferable to perform a simulation with the original datasets first, and also out of curiosity. The resulting graph showed interesting fractal structures. It seemed stranger than the usual strange attractors of the LM-63. This was done as follows.

The preparation of data was performed using the GIS system of the software. Then the resulting raster data yielded three sequences of DJF for the required 30 years of observations. Note that, because our parameters of causation are the net radiation flux and temperature (air) (in daily mean), the ground level was selected. This is also due to the time and space savings of the study. This can be the case for other future research. This selection of time interval is usual in climate studies, but as stated at the end of Chapter 3, it is also seriously subject to question: how can one know the relaxation time of the climate system to say that after 30 years of time interval — or even more or less — the system reaches its equilibrium point, even generally? However, for instance, in Hegerl et al. (2010), or identical research, an attempt was made to measure the return time of the climate system mainly by the CO₂ radiative forcing, which yields 25 up to 125 years, which has a great time uncertainty. The common approach to treating a deterministic model of ODEs is to implement numerical methods instead of analytical methods because of the practicalities mentioned in Chapter 3. The first experimental step was testing the model for all 10,000 days of the period on the 3-D Lorenz-63 system. For this purpose, the Runge–Kutta numerical method solved the system of equations in 4th order. The resulting graph of the model is as below (Fig. 3).

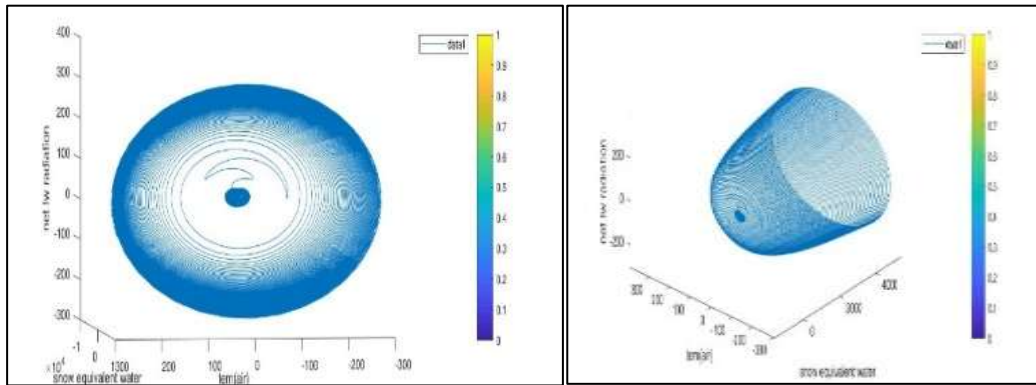


Fig. 3. The resulting realization of the LM-63 model obtained by implementing the Runge–Kutta method in 4th order.

Fig. 3. The x-axis corresponds to the mean temperature (K) (air) up to 2 meters above the ground (from -300 to $+300$ K), the y-axis belongs to the snow depth (inch), and the z-axis is for the net radiation flux (W/m^2) (from -300 to $+400$). However, with changes in the position of the values or even in their magnitude, there was no change in the shape of the fractal structure of the graph.

Fig.2 showed the primarily resulted graph of the 3-D lorenz-63 model for the 3 parameters of temperature (air), net radiation flux and snow equivalent water (snow depth wo). The time interval is equal to 10000 days of all the period of study. The chaotic behavior of the system is clear. Also, the exponential behavior after final deviation is made because of fluctuation of Lyapunov exponent. That is Lyapunov exponent identify the expansion of the stage attractor fractals. Also, it is showed that the exponent is positive i.e. $\lambda > 0$. The time evolution for this exponent is presented in the figure (4).

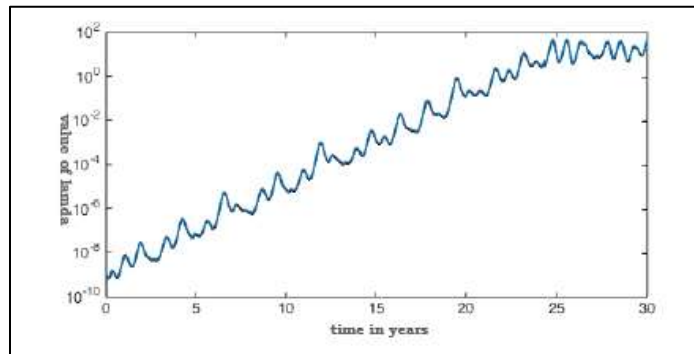


Fig. 4. the time evolution of Lyapunov exponents corresponding to the dynamical trajectories for 30 years of the whole period of the study

Fig. 4. the time evolution of Lyapunov exponents corresponding to the dynamical trajectories for 30 years of the whole period of the study which is evidence of the chaotic and exponential geometry of fractal structure of the simulated model of LM-63 by three initial conditions of the original data.

This guaranteed the chaotic behavior of the system and, consequently, of the parameters that defined the required vectors or sets of values. The $\Delta t = 1$ for each day of the whole period, which is therefore the number of time increments in the model. The resulting graph of Figure (2) illustrates a world without forcing and with only internal variability. Please note that this is just the experimental performance on the LM-63 model using the original set of data, while the LM-63 and SLM-63 are performed using the corresponding PDF (FPE) values.

The latter is based on the foundations of state-space models and discrete nonlinear models with additive white noise, along with "recursive Bayesian state estimation" as the most fundamental .

4.2. SLM-63 and LM-63 as Parallel Factual and Counterfactual Worlds

The parallel LM-63 gives us the counterfactual world, and solving it with the EnKF gives us an optimal solution for a large dynamical system such as the system in this study. From Figure (5), the fact of the growing amount of net radiation flux at the surface is clear. However, as shown in the graph, when the net radiation flux has a higher value, the snow precipitation and the mean temperature (air) experience their lowest values! This point can also be a topic for future research.

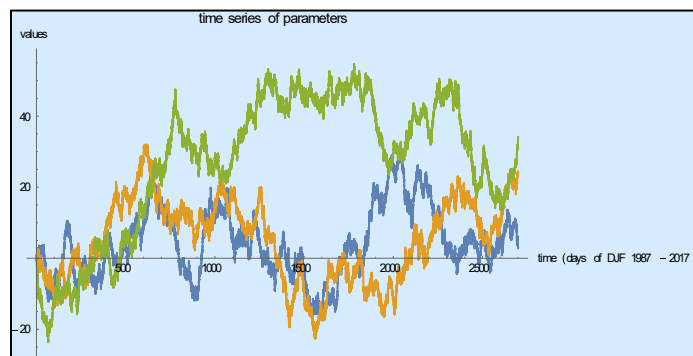


Fig.5. the time series distributions of original observations for three parameters in winter time (DJF).

Fig.5. The blue line is the time series distribution the snow precipitation (equivalent water of snow depth). The orange line for the mean temperature(air) and the green is for net radiation flux as the causing force for the extreme snow in the region of study. It is identified from the graph that however the primary days the three lines are moving along with each other in the middle of the period the radiation line diverge from the normal line and has very high positive value while the two others are experiencing a moving along path approximately.

There was a need to transform above distribution to a PDF distribution of three parameters. This is calculated by the kernel density smoothing since it treated as an integral transformation. The resulted graph is represented below figure (6).

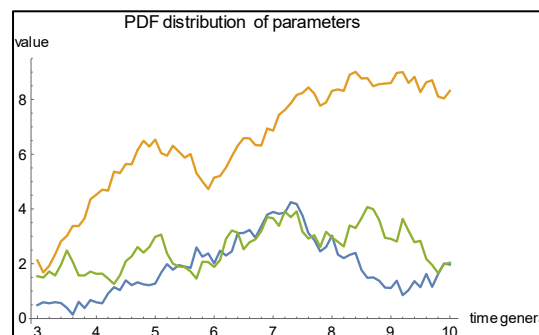


Fig. 6 PDF distribution of the three in used parameters

Fig. 6 PDF distribution of the three in used parameters i.e. blue line for snow depth equivalent water, the green line for net radiation flux and the orange line for mean

temperature (air). the initial conditions for the corresponding FPE of the model is derived from this distribution.

For the counterfactual part the LM-63 with the initial condition of PDF of observations (figure.5) is executed. Which showed little dense fractal structure that is because of the initial condition values which here were the values of PDF of observations (Fig.6). However, in the previous condition (Fig.3) with the changing in the place of the values or even the amount of them there was no change in the shape of fractal structure of the graph here with a change in value you have another fractal structure. For the factual world however, there was a need to add white noises to the original LM-63 to make it SLM-63. The white noises were with Gaussian distribution with zero mean and known deviation. The graph of the noises is as bellow, figure (Fig.7).

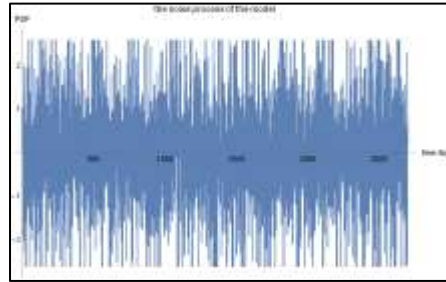


Fig. 7 - the graph of white noise process that has the role of making the perturbation for the SDEs of the model

Fig. 7 - the graph of white noise process that has the role of making the perturbation for the SDEs of the model. as it is clear the distribution of the noise process is a Gaussian distribution which is centered at (-1) to (+1). These noises are the correspond to the generation of random attractors in the SLM model.

The noises in the figure (7) are uncorrelated namely their covariance is equal to zero. This help to have a Gaussian distribution for the mutually exclusive noises. However, the nature of intensity of noises may or may not be Gaussian retribution but the results of the filter is fairly promising in this study and also in several identical cases . Although the reason of this success is not clear yet. To realize the counterfactual and factual corresponding PDF(FPE) the EnKF is implemented on the LM-63 with its unperturbed FPE and also on the perturbed SLM-63 as the counterfactual and factual worlds. In fact, these are the realization of ω over Ω and Ω' . Then are showing the probability density of three defined parameters in counterfactual and factual (left to right) worlds respectively. In this respect the graph can be interpreted as he resulted graphs of the model in the forecast/measurement state are as below in Figure (8).

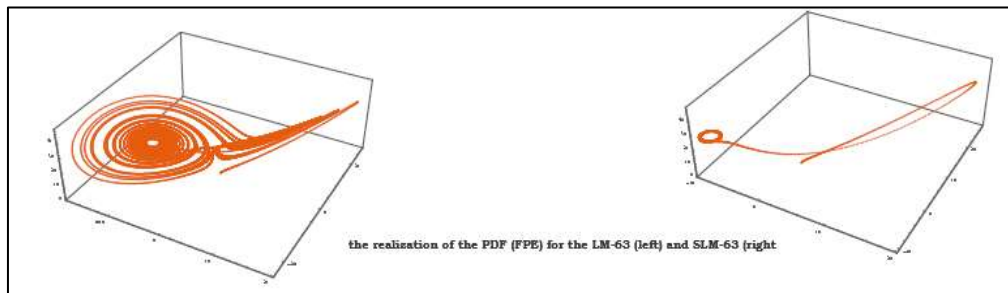


Fig 8. two graphs of the resulted LM-63 and SLM-63 left and right respectively for the forecast step.

Fig 8. two graphs of the resulted LM-63 and SLM-63 left and right respectively for the forecast step. These graphs are produced by the PDF(FPE) values of the X_t and y_t . the symmetric behavior then equilibrium manner of the counterfactual world is showed in the left graph and the non-symmetric behavior then non-equilibrium manner of the factual world is showed in the right.

IN this figure the less dense graph against the figure (3) is because of the truncated and transformed data as PDF as initial condition and also for the added noises in the factual world. This also caused for the predictability and less chaotic behavior of the factual model than counterfactual model. The important result of the graph then is the power of SDEs system of equation and also the Bayesian nonlinear filtering as EnKF. In fact, this is the forecast step of the ensemble Kalman filter.

Here the task was this; associate a label $\omega \in \Omega(w_t v_t)$. This derives the stochastic model in the context of EnKF and the related SLM-63. The role of these noises was to perturbed the observation or measurement in the SDEs system and propagate the system in its evolution interval. However, the time is implicitly implemented to the model by the model itself. The SDEs which produced this result is solved actually by the EnKF's intrinsic conditional probability and it roots in the Bayesian produced (chapter 3) random process. Here the strange attractors of the previous model with the original data turned to the random attractors which is perturbed and produced by the stationary white noises with the covariance given by identity matrix i.e. Q and R . The $(\delta_1, \delta_2, \delta_3)$ is identified by equal values of $\sqrt{1/2}$ as the intensity of forcing and $\sigma(v, w) = 0.5$. The significant point of these two figures is the fact that for the counterfactual world we encounter with the PDF in the equilibrium point and that showed that reaching to equilibrium and have internal variability and predictability with the current deterministic approaches is in the condition of having a system without external random perturbations while it is impossible. In figure of SLM-63 we can see the non-equilibrium nature of the PDF which by itself identified the dissipative, random, and also chaotic nature of the climate system in our general view. The forced ODEs system of the above deterministic chaotic model turned into SDEs system of equation by adding w_t and v_t of white noises. Here the task was this; associate a label $\omega \in \Omega(w_t v_t)$. This derives the stochastic model in the context of EnKF and the related SLM-63. The role of these noises was to perturbed the observation or measurement in the SDEs system. Also, it must be note that the SDEs system defined in this study is stationary .it means that the PDF (which here is the FPE actually) of the state vector i.e. X_t has no dependence on the initial condition after sufficiently long-time t namely initializing step at X_0 and also on the t by itself. This PDF (FPE) is the mean over Ω of the sample measure is calculated by each realization ω (solution in this context) of white noises mentioned above $(w_t v_t)$. The graph of the noise process for 2700 days is identified in the figure (7). About the SLM-63 system each sample measurement is supported by a random attracter instead of deterministic attractor or strange attractor of LM-63. This attractor has a fine structure (fractals) and was also dependent on time. But this PDF(FPE) of the unperturbed Lorenz-63 as showed in (Fig 3) was for counterfactual world has showed the strange attractor or Lorenz attractor smoothly.in the factual world however the nature of the PDF is identical but its fractal is affected by the forcing parameters.

4.3- Conclusions of the implementing EnKF for two parallel worlds

This step is the calculation of the PDF (FPE) for the models of LM-63 and SLM-63 corresponding to the counterfactual and factual worlds showing forecast step in the algorithm of EnKF in the context of Lorenz-63. The trajectories in the previous graphs are

made by initial conditions of PDFs of the three parameters of mean temperature, net radiation flux and snow equivalent water. The kernel density estimation ($K(x, t)$) is applied to the ensemble of simulation obtained from forcing to estimation of the corresponding PDFs. As mentioned in chapter 3 the whole process of SDEs analysis needs integration either analytical or numerical then there is need to integral equation and the transformations. The EnKF are doing this task here and yield numerical integration to the SDEs system here. These results are explained in the following by two trajectories in figure (9). these trajectories are showing the evolution of the simulation in the ensemble space through the observation time interval. Note that this evolution in time is corresponding to the LM-63 and SLM-63 as the counterfactual and factual worlds respectively.

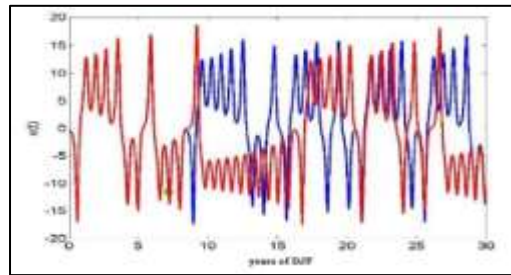


Fig. 9. the trajectories for the forecast step of LM-63 (red) and SLM-63(blue) in comparison with each other

Fig. 9. the trajectories for the forecast step of LM-63 (red) and SLM-63(blue) in comparison with each other. The time is in years to have smoother trajectory. The trajectories are meeting each other in just two trivial areas. This is identifying the difference between the LM-63 and SLM-63 in their time evolution.

Hitherto the forecast step or measurement step of the EnKF is implemented on the situation of the case study. To reach to the aim of this study there is a need to gain the causal probability. This would be fulfilled by the analysis step of the EnKF. As mentioned before this phase do this task as a maximum likelihood in a few sub-steps. First is combination of PDFs of the true states with the observation all in one. This is performed for all 2700 days of DJF (winter time) of the time interval of the study in its LM-63 and SLM-63 i.e. counterfactual and factual parallel worlds. The figure (10) displayed the results of the combination in two very dense figure of the three PDFs with the observation.

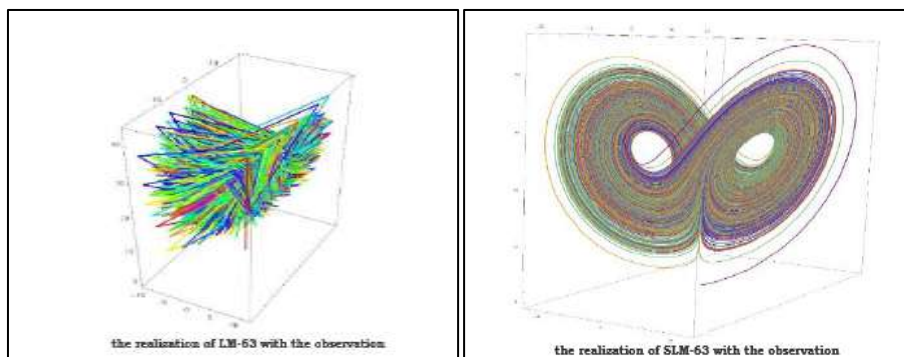


Fig. 10. the displayed figures of the analysis step of the EnKF for the LM-63 and SLM-63

Fig. 10. the displayed figures of the analysis step of the EnKF for the LM-63 and SLM-63 as the counterfactual and factual worlds left to right respectively. As it is clear from the left

figure the resulted graph for the LM-63 in the context of EnKF filtering showed a very dense but non-smoothing shape and irregular with an ambiguous iteration in its evolution in time.

In this figure although in the factual world, EnKF showed a smoothing result with a clear iteration of the dynamical system. this comparison also showed the fact that the white noises which added to the model in its SLM-63 part is the causing of this smoothing and symmetric analysis. The coarse appearance of the left figure is the evidence of unpredictability of the occurrence of the extreme snow fall in Mazandaran province in a deterministic world but predictability of the event in the world of stochastics. The smooth and symmetric appearance are the prominent evidence. Above all these are the deterministic shapes of the defined random process to the model.

Please here note that the logic behind these results and is interpretations for the aim of this study is on the basis " the fractal geometry of nature" by Mandelbrot (1982). So, may be called it ambitious aim of the future is on the Fractals representation of the dynamical system. which it seems that is a fairly certain road to solve the problem as it basically is on the chance and probability the readily displaying the noised in the phenomena that makes it unpredictable or so! The difference between these two PDF (FPE) generated graphs is that they are identifying the existence of an area of the phase space that is more likely to be reached in the factual than in the counterfactual world (Fig. 6). The note is that this result is due to the EnKF formulation which is illustrated in chapter 3 that is actually was a combination of the Monte-Carlo and Bayes' theorem. Then is initially and fundamentally random and probabilistic such as the climate by itself. According to algorithm of the EnKF the covariance matrix has an important role in the process of filtering. Since it is the identification of the amount and shape of the variations of the corresponding parameters in the phase space. Another reason is evidence of the efficiency of the process on this case study of extreme snowfall of Mazandaran province. Figure (11) is the proof of this expressions for the counterfactual and factual worlds as below.



Fig.11 the 3-D plot of the covariance matrix corresponding to the LM-63 (left) and SLM-63 (right)

Fig.11 the 3-D plot of the covariance matrix corresponding to the LM-63 (left) and SLM-63 (right) in the context of EnKF for the counterfactual and factual worlds respectively. The significance point of these figures is the distribution of the extreme observations in two worlds. This is shown that the SLM-63 has very more points than the LM-63.

Figure (11) is clear evidence of the Advantages of the stochastic process on the deterministic proceed for the extreme attribution study in the context of climate change. The symmetric distribution of the produced points in the analysis step of the Ensemble kalman filter along with the observation points is evidence of the efficiency of the SLM-63 model for the factual world than the LM-63 for the counterfactual world.

Then we are searching for sequences from both worlds in which an event did accrue, by identifying the leading direction of projections of the sequences in the phase space. Consequently, the goal is to identification of the maximum likelihood of the trajectory of the sequences in the phase space to be reached in the factual world than in the counterfactual one (if there existed). Therefore, it is clear that the probability of the event in the former is found to be higher than in the latter, namely $p_1 > p_0$. then the occurrence of an event in this context can be define mathematically as $\{\max_{\{0 \leq t \leq T\}} \delta' y_t \geq u\}$, where δ' is the unit vector of forcing. The resulted graph of this process is showed by the figure (12). The main interpretation of this graph is the existence of the random noise for the constant forcing in the SLM-63 that caused the more likelihood of extreme snow fall occurrence in this part of the model than in the part of LM-63. The first as the factual and the second as the counterfactual worlds.

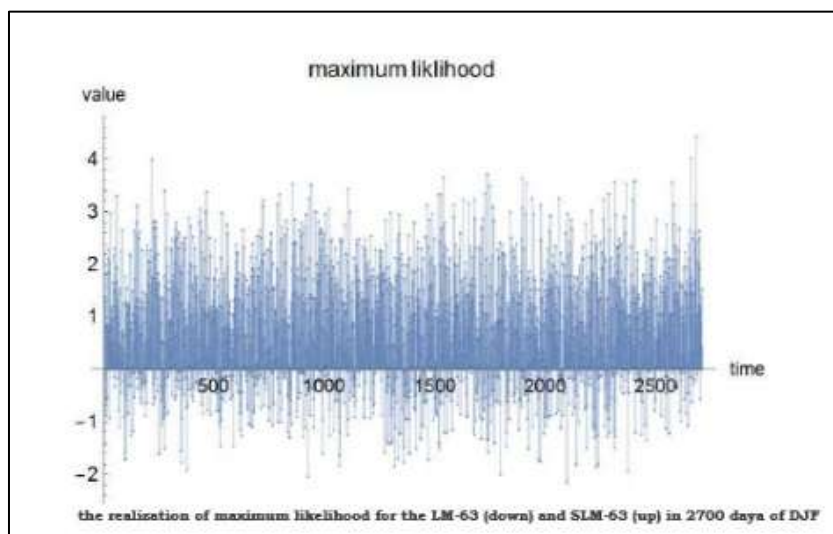


Fig.12. the time series of the values of maximum likelihood of the occurrence of extreme snow fall in Mazandaran province in factual (up) and counterfactual (down).

Fig.12. the time series of the values of maximum likelihood of the occurrence of extreme snow fall in Mazandaran province in factual (up) and counterfactual (down). The main interpretation of this graph is the existence of the random noise for the constant forcing in the SLM-63 that caused the more likelihood of extreme snow fall occurrence in this part of the model than in the part of LM-63. The first as the factual and the second as the counterfactual worlds. The amount of predictability and attributability to the forcing which is represented as the net radiation flux is also clear. This is actually a display of the log likelihood result of the proposed likelihood function in chapter three? More illustration in the text.

This is the enlightening form of the process because it from a causal perspective. And that is due to the fact that the related necessary and sufficient probabilities for the causation of the events are positive. Figure (4.9) is resulted from the trajectories associated with the event in two worlds appear to have distinct features. The positive part as the factual and the negative part as the counterfactual worlds. The amount of predictability and attributability to the forcing which is represented as the net radiation flux is also clear.

The likelihood (maximum) function is transformed of the probability for a single point in theoretical way. Here this is defined as ML_0 and ML_1 for the counterfactual and factual world which is the basis for the calculation of three required probabilities of necessity, sufficiency

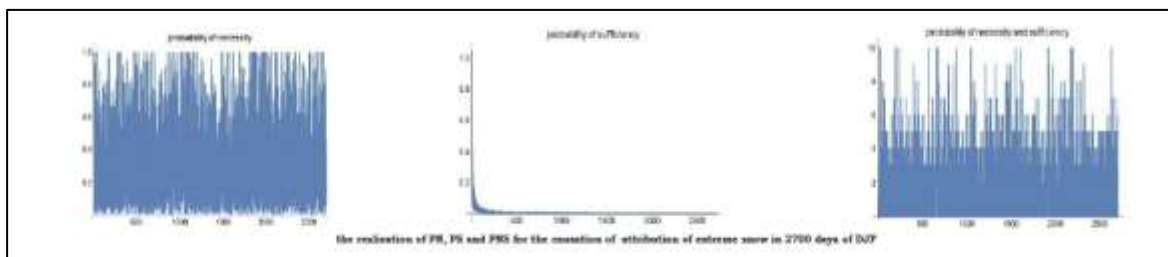
and necessity and sufficiency in the context of causal attribution theory. This is done by the help of Bayes theorem since the conditional relation between the three parameters of the study is the main foundation and also the end is the causal attribution to the forcing as the net radiation flux by the connection of mean temperature to the extreme snow fall. For clarification the end formulation is brought here as;

$$L = \frac{p(y|x)p(x)}{p(x|y)} \quad (4.1) \text{ The}$$

numerator is the model promise and denominator is the end result. This rooted in the theorem of total probability and is a inferential approach that concentrate on x. in fact taking the logarithm of the likelihood function in its Gaussian assumption yield this result such that the maximum likelihood of the Fig.9.4 is actually the log likelihood results of the proposed likelihood function. for more refer to the section 3. As illustrated in the previous sections the main theoretical concepts of this study are LM-63 and SLM-63 dynamical models which was selected to bring us the factual and counterfactual worlds. The resulted and depicted proposed figures of were going to do a compression in respect of these two near real time worlds. The critical part was the corresponding trajectories of the state vector x_t simulated under factual conditions namely in the presence of the additional forcing. This is also perturbed by the white noises to stochastically guarantee the required situation. Then this trajectory along with the observations y_t that is showed by the figure (6) in T=2700 (0f DJF days) are combined by the process of the EnKF to brought us the log likelihood of the occurrence of extreme snowfall with the defined forcing in the context of chaotic model of Lorenze-63.

The EnKF is used to assimilate these observations into a model of factual world $p=1$ consequently it matches the forecast/measurement model used in the simulation step then a reconstructed trajectory is obtained from corresponding analysis step. Also the Maximum likelihood which defined as $ML_{0,1}(y_t)$ to attribute to the state vector x_t . this is showed in the figure (8.4). Note that the required assimilation is done in the counterfactual and factual models to obtain corresponding trajectories, from the same observation namely one time with the SLM-63 and observation and one time with the LM-63 and observations. The corresponding likelihoods are LM_0 and LM_1 for the counterfactual and factual respectively. the figure (8.4) is illustrating this point. The local differences between the estimated likelihood of two trajectories of LM-63 and SLM-63 are showed in the figure (8.4). consequently, they are the instant inconsistencies between the associated factors of the ML (x, y). the cumulative difference is also identified by the figure which is a distinct gap between the results of the factual and counterfactual models.

The last step of the whole calculation for this study was the computation of the probability of occurrence with respect to data assimilation results (log likelihood) in the context of EnKF. This is done by the change in the Pearl (2000) definition of necessity and sufficient and necessity and sufficient conditions for the causation then attribution.



Figures of (13) is the probability graph of causation according to the PN, PS and PNS by their corresponding mean with a remarkable grows over time for the probability of necessity.

Fig.13. the computed probability of necessity, sufficiency and probability of necessity and sufficiency over the time interval of 2700 days of DJF from 1987-2017 for the extreme snow fall of Mazandaran province with respect to their corresponding forcing. These three graphs are showing the time distribution of the related probabilities over the time period. The probability of necessity possesses the greater values the probability of sufficiency has the most least values and the probability of necessity and sufficiency has the lesser values.

5. conclusion

The probability of sufficiency showed the most least values due to the fact that this probability according to the Pearl (2000) is the probability of causation without the causes that lead to the occurrence of the event in its probability of necessity. Therefore, in this case of study this is the probability of extreme snow fall in Mazandaran province without the defined forcing (net radiation flux as the forcing of all; as it is common in attribution studies). The probability of necessity is then the probability of extreme snow falls in Mazandaran province in the presence of the defined forcing. This probability with its greater values is as evidence of the whole calculations in this study along with the two others as its complement actually. The probability of the necessity and sufficiency is actually the probability of extreme snow fall in Mazandaran province with and without the defined forcing for this study. However, it still has very greater values than probability of sufficiency due to the great influence of the values of the probability of necessity. The mean values of theses probabilities are also computed in the table below.

Table.4.1 the mean values of the computed PN, PS, and PNS.

PN	PS	PNS
0.8978	0.1942	0.4519

Table.4.1 the mean values of the computed PN, PS, and PNS. These probability values are doing so as the p-value in the conventional approaches of Attribution of climate change. The meaning of the table (4.1) is that the extreme snow falls of Mazandaran province with the probability of necessity value of 0.8978 is attributable to the fluctuation of the intensity of the defined forcing (net radiation flux) in relation with the mean temperature (air) for the period of this study. It also is attributable to the other forcing in its value of probability of sufficiency in the value of 0.1942 and to its value of probability of necessity and sufficiency in the value of 0.4567.

These all together is the end result of the computation for this case of study about the attribution of extreme snow fall in Mazandaran province. So then according to this study the extreme snow falls of Mazandaran province in the 30 years of 1987 -2017 of DJF (winter time) is attributable to the fluctuation of the intensity of net radiation flux that caused the fluctuation of the intensity of mean temperature (air) in the general view of this study with its experimental but modified dynamical model of Lorenz-63 in the context of the EnKF.

This study had two main purposes. The first one was the practice of the illustrated method instead of complex climate models in real case with a general view of parameters, physical, mathematical approaches. However, the cause and casual theory of the Pearl (2000) is the main thread which gather all of them. And the second one was the a more ambitious aim and that is also for now and future. The EnKF was implemented as data assimilation technique to provide the required maximum likelihood based on the mentioned PDFs. This study was trying to find causal links between the change in net radiation flux as climate forcing (forcing of all) and the change in mean temperature (air) with the extreme snow fall

of Mazandaran province. This is called detection and attribution in climate studies as well. Therefore, it searched for the probability change of the occurrence of extreme snow fall in Mazandaran province with change in mentioned forcing i.e. net radiation flux that caused the change in mean temperature (air) and that caused change in probability of extreme snow fall in the region.

5.1 Discussion and suggestions

From the illustrated points and theoretical basis of this study it can be interpreted that the randomness and noisy behavior of the climate system is clear and it is clear that because of the complexity of the system not the dynamic and not the physics of the system can be illustrated by the conventional deterministic methods as the climate system is an open and extended system not a closed system and easily prone to noises. So then the approach which can be helpful in practice is entering to the world of fractal geometry since it is rooted on probability not on the determinism and more importantly provide an iterated but random shapes. This by itself explain the fact that the causes are there but the intensity is differing randomly. Then one of the main tools which is very practical in this context is the development of the numerical solutions of stochastic differential equations instead of ODEs or PDEs. SDEs are the extension of ODEs in the realm of noises. In fact, the errors do not calculate separately but is a part of the system. According to the Brownian motion these noises are random sense the motion is random as well in a microstate scale. Then According to the butterfly effect theory and cause' and also causality the randomness is a natural and inseparable part of any natural system.

Climate system is a natural system it must be random and governed by probability. Then we are faced with a stochastic not deterministic system. Then we need to move on the house of SDEs and random process instead of ODEs. According to the Mandelbrot (1982) most of the nature is irregular and random so the geometry of it is fractals not the Euclid. The simplest and more practiced model for the start of a novice researcher which has this property and the property of the last paragraphs is the Lorenz-63 chaotic dynamical model.

In fact, in this study the motivation of the usage of this model for a long time of period is that the SDE system of it is also practiced in literature and plenty of paper and books worked on it even the Mandelbrot in his book of "fractal geometry of nature". According to the above discussions walking on the path of fractals is the way of success for climate modeling which also is a low cost approach. So the main task is developing numerical methods for solving SDEs in a large-scale system such as climate. Subsequently, if you can display the suitable fractal of the case study in this context you are home. The main method presented here were ODEs system vs. SDE system as the factual and counterfactual worlds. The reason for this parallel selection of worlds instead of having a long time period such as 100 years is that the real presented data of climate system is not so long as this and the Brownian motion in the system do not allow to an exact or near production of UNSEEN data! Another reason is that weather forecasting in current models is on the 5 days of certainty then how the climate models in current generations can predict the far future! Then period of the study is separated to two parallel words of 30 years (DJF of 1987-2017) in near real time with certain near real data.

Such as statistical mechanics the ensembles concept is used. Then the probability of occurrence of extreme snow in an ensemble of simulation by LM-63 and SLM-63 models. These models were representing the observed climate conditions which simulates the probability of occurrence of extreme snow fall in the real world i.e. SLM-63 with the probability of occurrence of the same event in a parallel world of ensemble of its simulations i.e. LM-63 as an alternative. In literature these worlds dubbed as factual and counterfactual

worlds. If p_1 and p_0 be the probability of occurrences in the factual and counterfactual world respectively. Then the probability of necessity, the probability of sufficiency and probability of necessity and sufficient for the causation of extreme snow fall of Mazandaran province can be derived as Pearl (2000). Although in this study this is turned to the likelihood instead of probability in the context of EnKF as the data assimilation technique. This part is dedicated to the future enhancement of the whole process in general and the method of this study in particular. However, this is actually not possible to overcome all the area, a brief review would be helpful.

Climate change research and its detection and attribution for the extreme weather and climate events is in progress. However, most of the current research is on the mere statistical method developments this study argued that with these approaches such as fingerprint methods or storyline methods which are regression based and frequency-based methods a great part of the problem left unseen! That is there is need to better understanding and modeling of the physics of the process. Although it is clear that understanding the whole process of the climate system is very time consuming and may be impossible but working on the classical theories and expect a modern result seems strange! Event with the aid of super computers. Therefore, as the modern physics declare the non-equilibrium and entropy increase of the whole universe, the climate system as a tiny part of this whole is sharing this property. Then we are dealing with a non-equilibrium, dissipative, chaotic, random and probabilistic system which uncertainty is a part of this system not something in outer space. Then as in it local and global view this study shown that the stochastic approaches and developing the numerical methods along with a serious attention to the "fractal geometry of nature" and climate as a part of it we can consider the problem, in a sound way. However, the significant case study of Lyapunov exponents is an interesting field. Although it is more placed in the deterministic world of computation, its combination with probability case would have more interesting result for the development of stochastic climate theory and models.

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